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ASYMPTOTIC BEHAVIOR OF THE FAR REGION OF TURBULENT WAKE VORTICES

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An approximate mathematical model, formulation of the problem, and its approximate solution are proposed for the far region of a turbulent vortex wake past a moving body, where the departure of the horizontal velocity component from the uniform flow is slight. It is assumed that the single important parameter that defines the main flow characteristics in this region is the vortex momentum per unit length produced in the fluid by the lift equal to the weight of the moving body uncompensated by the buoyancy force. Thus, the flow is self-similar, and the self-similarity law determines the intensity, shape, and location of vortex lines as functions of the downstream distance with accuracy up to a constant factor, which cannot be determined theoretically and should be obtained by comparison of theory with experiment. A boundary-value problem is formulated to determine the flow structure of vortex lines (vorticity distribution). A solution of the problem is obtained numerically in the limit of "vanishing turbulent viscosity." The variation in the maximum velocity of a vortex line with distance, determined by self-similarity, is in agreement with available experimental data.

Introduction. The problem of wake vortices has become especially pressing with the appearance of heavy airplane, behind which intense vortex lines extend over several kilometers and are a serious hazard to small airplanes falling in the region of these vortices [1].

In this connection, it is of interest to study the far region in a vortex wake, i.e., the region where the ordinary momentum-free wake (the drag force is compensated by the engine thrust) is no longer significant and, at the same time, the phenomena caused by the instability and breakup of a vortex pair into structures such as vortex rings, etc., are not yet manifested.

Experimental studies and theoretical descriptions of this phenomenon involve solving complex problems of the dynamics of concentrated vortices. Under laboratory conditions, measurements in the far region are impossible because of the limited dimensions of experimental installations and the significant effect of the wake due to the body drag (under real conditions, it is compensated by engine operation). Full-scale experiments are technically difficult and are complicated by many additional factors: atmospheric turbulence, wind, stratification of the atmosphere, etc. These circumstances lead to a wide spread of measurement results and, in some cases, to contradictory results.

Under real conditions, the motion in such vortices is turbulent, as confirmed in experiments. The lack of a reliable, serviceable mathematical model for describing turbulent fluid flow hinders the development of an adequate mathematical model and a fairly comprehensive theoretical description of this phenomenon.

The complexity of the problem is responsible for the considerable simplifications used in analytical studies. In a number of papers, one vortex line is considered, the flow in its vicinity is considered axisymmetric, and the effect of the second vortex line is ignored [2, 3]. This approach is justified on the initial segment after roll-up of the vortex wake shed from the wing (lifting surface) and as long as the variation in the total circulation due to turbulent diffusion of vorticity is negligible. It is clear that at a certain distance, the

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influence of vortices on each other becomes significant, the flow in the vicinity of each vortex ceases to be axisymmetric, and circulation decreases with time.

The approach used in the given paper is similar to the one used previously in studies of a turbulent vortex ring and a vortex pair (a plane analog of a vortex ring) [4–6]. The boundary-value problem for determining the flow structure considered in the present paper coincides with the corresponding problem for a plane analog of a vortex ring. It is assumed that there is no stratification of the atmosphere.

1. Equations of Motion. The equations for the mean flow describing the steady turbulent flow past a moving body in Cartesian coordinates in the conventional notation have the form

$$
\begin{aligned}
&W_0\frac{\partial u}{\partial z}+w\frac{\partial u}{\partial z}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{\partial\sigma_{xz}}{\partial z}+\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y},\\
&W_0\frac{\partial v}{\partial z}+w\frac{\partial v}{\partial z}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{\partial\sigma_{yz}}{\partial z}+\frac{\partial\sigma_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y},\\
&W_0\frac{\partial w}{\partial z}+w\frac{\partial w}{\partial z}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial z}=\frac{\partial\sigma_{zz}}{\partial z}+\frac{\partial\sigma_{zx}}{\partial x}+\frac{\partial\sigma_{zy}}{\partial y};
\end{aligned}
\tag{1}
$$

$$
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,
\tag{2}
$$

where W_0 is the velocity of the flow moving along the z axis from infinity, $v_z = W_0 + w$, and σ_{ik} is the tensor of viscous and turbulent Reynolds stresses. The x axis is directed along the gravity force.

At rather large distances downstream from the moving body, the velocity component w of the disturbance introduced into the flow is small compared to W_0, and the rate of change of the quantities with distance along the z axis is small, so that $\partial/\partial z \approx \varepsilon(\partial/\partial x) \approx \varepsilon(\partial/\partial y)$ ($\varepsilon \ll 1$), but $W_0(\partial/\partial z) \approx u(\partial/\partial x)$, $v(\partial/\partial y)$. Therefore, as a first approximation, the terms containing the derivatives with respect to z can be dropped, except for the first terms in Eqs. (1). It is convenient to introduce a fictitious "time" — the quantity $\tau = z/W_0$.

Below, we describe the turbulent flow using a simple model with a turbulent viscosity coefficient dependent on the "time" τ, so that, ignoring molecular viscosity, we can write the stress tensor as

$$
\sigma_{ik}=-\frac{1}{3}(u_k')^2\delta_{ik}+\nu_*(\tau)\Big(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\Big),
\tag{3}
$$

where u_i' are the pulsation velocity components and $(u_1, u_2, u_3) = (u, v, w)$.

With allowance for (3), Eqs. (1) and (2), as a first approximation, are written as

$$
\begin{aligned}
&\frac{\partial u}{\partial\tau}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p'}{\partial x}=\nu_*(\tau)\Big(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\Big),\\
&\frac{\partial v}{\partial\tau}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p'}{\partial y}=\nu_*(\tau)\Big(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\Big),\\
&\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,
\end{aligned}
\tag{4}
$$

where $p' = p + (1/3)\rho(u_k')^2$ is the modified pressure.

From Eqs. (3) and (4) it follows that as a first approximation, the velocity components perpendicular to the free stream does not depend on the longitudinal velocity perturbation. After u, v, and p' are obtained, w is determined from the equation

$$
\frac{\partial w}{\partial\tau}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+\frac{1}{\rho}\frac{\partial p'}{\partial z}=\nu_*(\tau)\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\Big).
\tag{5}
$$

From Eq. (5), the longitudinal velocity perturbation can be determined as is done in [7]. In the present paper, this problem is not considered.

2. Self-Similarity. Let the motion of a body at constant speed be maintained by engine operation. Behind the body, a momentum free wake forms (the total horizontal component of the forces acting on the body is equal to zero) in which the horizontal velocity component w decays rather rapidly. Below, it is assumed that at a certain distance downstream from the moving body there is a region in which the vortex wake related to the lift plays a leading role.

Let the weight of the moving body P be balanced by the lift. For compensation of the weight of the body, the fluid should be imparted momentum per unit time $J = P$. In time t, the body travels distance $z = W_0 t$, where W_0 is the speed of the body, and, hence, the momentum per unit length of the distance traveled is $J_0 = P/W_0$ or the vortex momentum is $2j_0 = J_0/\rho$. Here ρ is the density of the fluid, which is considered incompressible and homogeneous, and, with allowance for the flow antisymmetry about the plane $y = 0$, the vortex momentum is given by the formula

$$j = \int_{-\infty}^{\infty} \int_{0}^{\infty} y\omega_z \, dx \, dy.$$

This quantity is an integral of motion of system (4), and, thus, each section z = const contains constant (independent of the "time" τ) momentum $2j_0$. The dimension of the vortex momentum is $[j_0] = L^3/T$. If, with allowance for the turbulent pattern of the wake flow, the kinematic viscosity of the fluid is ignored, the vortex momentum is the only dimensional constant that should determine all flow characteristics. Under this assumption, the problem considered becomes equivalent to the problem of motion of a planar analog of a vortex ring — a vortex pair, produced in a fluid by introducing a vortex momentum in an infinitesimal fluid volume [4]. In this case, the quantity $\tau = z/W_0$ acts as the time. According to this, from dimensional considerations, we obtain the shape and location of the vortex lines:

$$x = x_0(P/2\rho W_0^2)^{1/3} z^{1/3}, \qquad y = y_0(P/2\rho W_0^2)^{1/3} z^{1/3}. \tag{6}$$

Here x_0 and y_0 are certain constants, determined by the position of the maximum of vorticity in the plane x, y for a vortex lines for which $y > 0$. If the flow pattern in the plane x, y is projected for different values of z onto the plane $z = 0$, from (6) we obtain a vortex pair moving with the "time" τ according to the above-mentioned analogy. In this plane, the trajectories of the projections of the points corresponding to each of the vortex lines are straight lines that issue from a certain virtual origin and pass (approximately) through the tips of the lifting wings, so that they form angle 2β. We set $\alpha = \tan\beta = y_0/x_0$. This quantity cannot be found (in the present formulation) theoretically and is related, as will be clear below, to the turbulent viscosity coefficient. By analogy with vortex rings, one might expect that α is a small quantity ($\approx 10^{-1}$–10^{-2}). Let $2b$ be the wing span (the span of the lifting surface). Self-similarity implies that in the projection onto the plane $z = z_s$, where the wing is located ($z = 0$ corresponds to the position of the virtual origin of the vortex lines), the distance between the vortex lines varies as $y = b + \alpha(x - x_s)$, where x_s is the coordinate of the lifting surface (wing).

According to self-similarity, the velocity components perpendicular to the free stream and the modified pressure are defined by the relations

$$u, v \sim W_0(P/2\rho W_0^2)^{1/3}/z^{2/3}, \qquad p' \sim \rho W_0^2 (P/2\rho W_0^2)^{2/3}/z^{4/3}. \tag{7}$$

From (5) and (7) it follows that $w \sim W_0(P/2\rho W_0^2)^{2/3}/z^{4/3}$. Downstream, the intensity of the vortex lines varies as $\Gamma = W_0(P/2\rho W_0^2)^{2/3}/z^{1/3}$.

The turbulent viscosity coefficient according to self-similarity is determined by the equality

$$\nu_* = \lambda \frac{j_0^{2/3}}{\tau^{1/3}} = \lambda W_0 \frac{(P/2\rho W_0^2)^{2/3}}{z^{1/3}}, \tag{8}$$

where λ is a dimensionless constant, which remains a free parameter in the formulation considered.

The self-similar variables x' and y' are given by the equalities

$$x' = \frac{x}{(j_0/W_0)^{1/3} z^{1/3}} = \frac{x}{j_0^{1/3}\tau^{1/3}}, \qquad y' = \frac{y}{(j_0/W_0)^{1/3} z^{1/3}} = \frac{y}{j_0^{1/3}\tau^{1/3}}. \tag{9}$$

3. Formulation of the Boundary-Value Problem. We convert from the required functions u and v to the new functions — vorticity $\omega = (\boldsymbol{\omega})_z$ ($\boldsymbol{\omega} = \operatorname{rot} \boldsymbol{u}$) and the stream function ψ

$$u = \frac{\partial \psi}{\partial x}, \qquad v = \frac{\partial \psi}{\partial y}.$$

Then, with allowance for (8), taking into account that, by virtue of self-similarity and (9),

$$\omega = \frac{1}{\tau}\,\omega'(x', y'), \qquad \psi = \frac{j_0^{1/3}}{\tau^{1/3}}\,\psi'(x', y'),$$

we obtain the following boundary-value problem (the primes are omitted). It is required to find a solution of the equations

$$\lambda\Big(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\Big) + \frac{1}{3}\,x\,\frac{\partial \omega}{\partial x} + \frac{1}{3}\,y\,\frac{\partial \omega}{\partial y} + \omega = \frac{\partial \omega}{\partial x}\,\frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y}\,\frac{\partial \psi}{\partial x}, \tag{10}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega, \tag{11}$$

subject to the boundary conditions

$$\omega = \psi = 0 \ \text{ for } \ y = 0; \quad \omega \to 0, \ \psi \to 0 \ \text{ for } \ x^2 + y^2 \ \to \ \infty \tag{12}$$

and the following normalization condition, which follows from the law of conservation of vortex momentum:

$$\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} y\omega(x, y)\,dx\,dy = 1. \tag{13}$$

The problem formulated above allows one to determine the vorticity-distribution pattern in the vortex pair and the flow produced by it. In this formulation, the entire flow is determined by one quantity — the coefficient λ. The quantity α is uniquely related to the latter and can be determined from experiments. This relation is given by $\alpha = y_0(\lambda)/x_0(\lambda)$, where $x_0(\lambda)$ and $y_0(\lambda)$ are the coordinates that determine the position of the maximum $\omega(x, y)$ in the plane (x, y).

Actually, the flow produced by the vortex pair generated by flow past a moving body is not self-similar. In the immediate proximity of the body, the vortex pair pattern depends on the details of the body shape. However, as in the case of vortex rings [4–6], one might expect that at some distance away from the body, the pattern of the vortex pair and the flow produced by it acquires a certain universal nature. The fluid flow and the vortex pair pattern in this region do not depend on the details of the shape of the moving body. Full information on the detailed mechanism of formation of the vortex pair in the flow past the body at long distances is contained in just one quantity — the coefficient α. After the vortex pair travels a distance of about several wing spans along the y coordinate, the action of turbulent viscosity leads to a self-similar vorticity distribution, which depends on the conditions of formation of the vortex pair on the initial segment only via the constant α.

4. Approximate Solution. An exact analytical solution of the formulated problem has not been obtained. For large values of λ, the existence and uniqueness of a solution is rigorously proved by Pukhnachev [8]. It is of interest to compare the solutions of the problem for small values of λ with experiments [this will be clear when we obtain the relationship (approximate) between λ and α]. According to this, the coefficient λ of the high-order derivatives entering the equations is a small parameter. This complicates the problem since flow calculations by numerical methods require (in the present problem) a fairly accurate determination of the positions of the vorticity extrema in the plane (x, y). In this connection, it is necessary and useful to study a model problem that possesses the main properties of the exact problem and, at the same time, is simple so that the solution can be brought to completion in analytic form.

Instead of system (10), (11), we consider the system

$$\lambda\Delta\omega + \frac{1}{3}x\omega_x + \frac{1}{3}y\omega_y + \omega = \psi(x_0, y_0)\omega_x; \tag{14}$$

$$\Delta\psi = -\omega \tag{15}$$

with the same boundary conditions (12) and normalization (13) as for system (10), (11). Here x_0 and y_0 are the coordinates of the point at which $\omega(x, y)$ reaches a maximum (by virtue of the apparent oddness of the solution for y, it suffices to consider only the upper half plane).

The model system (14), (15) can be regarded as a crude approximation of the exact system (10), (11). Equation (14) differs little from the corresponding Eq. (10) in the vicinity of the point (x_0, y_0) of the maximum $\omega(x, y)$ because at this point, $\psi_0(x_0, y_0) \sim \alpha\psi_y(x_0, y_0)$ according to self-similarity, and the second term on the right side of (10) can be ignored compared to the first since α is small, as indicated above. At large distances from the point (x_0, y_0), as $x^2 + y^2 \to \infty$ the linear (left-hand) side, which is the same in both equations, becomes principal. According to this, one might expect that the solution of the model problem would give even the order of magnitude of the position of and value of the maximum $\omega(x, y)$.

The model problem is nonlinear as is the exact problem since the coefficient at ω_x on the right side of (14) is a functional of $\omega(x, y)$.

A solution of the model problem can be obtained as follows. We make the substitution of variables:

$$\xi = \frac{1}{\sqrt{6\lambda}}[x - 3\psi_y(x_0, y_0)], \qquad \eta = \frac{1}{\sqrt{6\lambda}}y, \qquad \omega(\xi, \eta) = \exp[-(1/2)(\xi^2 + \eta^2)]Z(\xi, \eta). \tag{16}$$

As a result of the substitution (16), from Eq. (14) we obtain the following equation for $Z(\xi, \eta)$:

$$Z_{\xi\xi} + Z_{\eta\eta} + (4 - \xi^2 - \eta^2)Z = 0. \tag{17}$$

Equation (17) admits separation of variables. Assuming that $Z = U(\xi)V(\eta)$, for U and V we obtain the equations

$$U'' - \xi^2 U = -(C + 4)U; \tag{18}$$

$$V'' - \eta^2 V = CV, \tag{19}$$

where C is the constant of separation, with the boundary conditions

$$U \to 0 \text{ as } |\xi| \to \infty, \qquad V \to 0 \text{ as } |\eta| \to \infty. \tag{20}$$

In addition, by virtue of the antisymmetry condition, the following condition should be satisfied:

$$V(\eta) = V(-\eta). \tag{21}$$

Thus, we have two eigenvalue problems related to one another by the constant of separation C. The eigenvalue problem for equations of this type has been adequately studied and its solution is known. Solutions of Eqs. (18) and (19) subject to the boundary conditions (20) and the antisymmetry condition (21) exist only if

$$C + 4 = 2n + 1 \quad (n = 0, 1, 2, \ldots); \qquad -C = 2m + 1 \quad (m = 0, 1, 2, \ldots). \tag{22}$$

From (22) it follows that $n = 0$ and $m = 1$ or $n = 1$ and $m = 0$. The antisymmetry condition for η leaves just one possibility $n = 0$ and $m = 1$. The corresponding solution has the form $U(\xi) = A_1 \exp[-(1/2)\xi^2]$, $V(\eta) = A_2\eta \exp[-(1/2)\eta^2]$. Hence, $Z(\xi, \eta) = A\eta \exp[-(1/2)(\xi^2 + \eta^2)]$, where A is an arbitrary constant determined by the normalization (13) and is equal to

$$A = \frac{\sqrt{6}}{9\pi\lambda^{3/2}}. \tag{23}$$

Equation (15) defines $\psi(x, y)$ via the already known $\omega(x, y)$. Taking into account that the stream function must be equal to zero at infinity, we have

$$\psi(x, y) = -\frac{1}{4\pi}\int\int_{-\infty}^{\infty} \omega(x', y') \ln[(x - x')^2 + (y - y')^2]\, dy'\, dx'.$$

However, the function ψ is easier to calculate as follows. In the variables ξ and η, Eq. (15) has the form

$$\psi_{\xi\xi} + \psi_{\eta\eta} = -6\lambda A\eta \exp(-\rho^2), \tag{24}$$

where $\rho^2 = \xi^2 + \eta^2$. We seek $\psi(\xi, \eta)$ in the form $\psi(\xi, \eta) = \eta f(\rho)$. Substituting this into (24), we obtain the following ordinary differential equation of the second order for $f(\rho)$:

$$\rho\Big(\frac{1}{\rho} f'\Big)' + \frac{4}{\rho} f' = -6\lambda A \exp(-\rho^2).$$

The general solution of this equation is

$$f(\rho) = C_1 + \frac{C_2}{\rho^2} - \frac{3}{2}\frac{\lambda A}{\rho^2}\exp(-\rho^2).$$

The arbitrary constants are determined from the condition of equality to zero at infinity and limitedness at zero. As a result, we have

$$\psi = \frac{3}{2}\frac{\lambda A}{\rho^2}(1 - \exp(-\rho^2)).$$

Reverting to the variables x and y, determining the values of x_0 and y_0, and taking (23) into account, we ultimately obtain the solution

$$\omega = \frac{y}{9\pi\lambda^2}\exp\Big[-\frac{(x-x_0)^2 + y^2}{6\lambda}\Big], \qquad \psi = \frac{1}{\pi}\frac{y}{(x-x_0)^2 + y^2}\Big\{1 - \exp\Big[-\frac{(x-x_0)^2 + y^2}{6\lambda}\Big]\Big\},$$

where

$$x_0 = \frac{1}{\pi\lambda}(2\exp(-1/2) - 1) = \frac{0.0686}{\lambda}, \qquad y_0 = \sqrt{3\lambda}, \qquad \alpha = \frac{\pi\sqrt{3}}{2\exp(-1/2) - 1}\lambda^{3/2} = 25.54\lambda^{3/2}.$$

The value of $\omega(x_0, y_0)$ at the point where it reaches a maximum is

$$\omega_{\max} = \frac{1}{3\pi\sqrt{3e}}\frac{1}{\lambda^{3/2}} = \frac{0.0372}{\lambda^{3/2}}.$$

The position of the maximum $\psi(x_1, y_1)$ is specified by the equalities $x_1 = x_0$ and $y_1 = \beta y_0$, where the value of β ($\beta \neq 0$) is obtained from the equation $1 + \beta^2 = \exp(\beta^2/2)$, where $\beta = 1.585$. The value of $\psi(x, y)$ at this point is

$$\psi_{\max} = \frac{1}{\beta\pi\sqrt{3\lambda}}(1 - \exp(-\beta^2/2)) = \frac{0.0829}{\sqrt{\lambda}}.$$

Along with the stream function ψ in a coordinate system in which the fluid is at rest at infinity, it is possible to consider the stream function Ψ related to ψ by $\Psi = -(1/3)x_0 y + \psi$. This function describes the flow in a coordinate system moving together with the vortex. In this system, the boundary $\Psi(x, y) = 0$ divides the flow plane into two regions: an outer region, in which the streamlines go from infinity to infinity, and an inner region, in which the streamlines are closed. It is easy to see that this boundary is a circle $(x - x_0)^2 + y^2 = a^2$, where the radius of the circle $a = \beta_1 y_0$ ($\beta_1 \neq 0$) is determined by the equation

$$(2\exp(1/2) - 1)\beta_1^2 = 1 - \exp(-\beta_1^2/2), \quad \beta_1 = 2.02, \quad a = 3.50\sqrt{\lambda}.$$

In this coordinate system, the position of $\Psi(x_2, y_2)$ is specified by the equalities $x_2 = x_0$ and $y_2 = \beta_2 y_0$, where β_2 ($\beta_2 \neq 0$) is obtained from the equation

$$1 + (2\exp(-1/2) - 1)\beta_2^2 = (1 + \beta_2^2)\exp(-\beta_2^2/2) \qquad (\beta_2 = 1).$$

One might expect that the orders of magnitudes obtained in the solution of the model problem coincide with the orders of magnitudes that must be obtained in the solution of the initial problem, at least for $\lambda \to 0$. Assuming that this is true, we introduce new variables and required functions by the equalities

$$\xi = \lambda^{-1/2}(x - \lambda^{-1}\xi_0), \quad \eta = \lambda^{-1/2}y, \quad \xi_0 = \lambda x_0, \quad \omega = \lambda^{-3/2}\omega_*(\xi, \eta), \quad \psi = \lambda^{-1/2}\psi_*(\xi, \eta).$$

After this substitution, system (10), (11) takes the following form (the asterisks at $\omega_*(\xi,\eta)$ and $\psi_*(\xi,\eta)$ are omitted):

$$\frac{\partial^2\omega}{\partial\xi^2}+\frac{\partial^2\omega}{\partial\eta^2}+\frac{1}{3}\xi\frac{\partial\omega}{\partial\xi}+\frac{1}{3}\eta\frac{\partial\omega}{\partial\eta}+\omega=\mu\Big[\frac{\partial\omega}{\partial\xi}\Big(\frac{\partial\psi}{\partial\eta}-\frac{1}{3}\xi_0\Big)-\frac{\partial\omega}{\partial\eta}\frac{\partial\psi}{\partial\xi}\Big]; \tag{25}$$

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2}=-\omega,\qquad \mu=\lambda^{-3/2}. \tag{26}$$

The boundary conditions and normalization take the form

$$\begin{gathered}\omega=\psi=0 \quad \text{as} \quad \eta=0;\qquad \omega\to 0,\quad \psi\to 0 \quad \text{as} \quad \xi^2+\eta^2\to\infty;\\ \int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\eta\omega(\xi,\eta)\,d\xi\,d\eta=1.\end{gathered} \tag{27}$$

In the new variables, all characteristic quantities — the maxima ω and ψ, the quantity $\eta_0 = \lambda^{-1/2}y_0$ which is the radius of the vortex, and $\xi_0=\lambda x_0$ — remain finite as $\lambda\to 0$. The parameter $\mu=\lambda^{-3/2}$, which enters Eq. (25), plays the role of the Reynolds number in the problem considered.

We set $\psi=\Psi+(1/3)\xi_0\eta$. With this substitution, the boundary conditions for Ψ at infinity take the form

$$\Psi_\eta\to-\frac{1}{3}\xi_0,\quad \Psi_\xi\to 0 \quad \text{as} \quad \xi^2+\eta^2\to\infty.$$

This substitution corresponds to conversion to a coordinate system attached to the vortex, and the value of ξ_0 is determined from the condition that the maximum ω is on the curve $\xi=0$.

5. Passage to the Limit $\lambda\to 0$ ("Vanishing Viscosity"). The approximate solution obtained above suggests that in the new variables, as $\lambda\to 0$ (accordingly, $\mu\to\infty$), the solution tends to a certain limiting solution. Under this assumption, it follows from (25) that in the limit, $\mu\to\infty$ the relation $\omega=\Omega(\Psi)$ holds but the form of the functional dependence $\Omega(\Psi)$ remains uncertain.

In the required flow, the streamline $\Psi=0$ divides the flow region into two regions: an outer region, in which the streamlines go from infinity to infinity, and an inner region (the atmosphere of the vortex pair), in which the streamlines are closed. By virtue of the boundary condition for ω at infinity, it follows from (27) that in the outer region $\Omega\equiv 0$ in the limit. The form of $\Omega(\Psi)$ in the inner region is to be determined.

A similar problem was considered in [6] for a turbulent vortex pair (and a vortex ring). We integrate Eq. (25) over a region with the boundary defined by a certain closed streamline. It is easy to verify that the right side identically vanishes, and, as a result, for any μ, we have the equality

$$\oint\nabla\Omega\boldsymbol{n}\,dl+1/3\oint\xi\Omega n_\xi\,dl+1/3\oint\eta\Omega n_\eta\,dl+1/3\iint\Omega\,d\xi\,d\eta=0. \tag{28}$$

Here dl is an element of the length of the streamline and $\boldsymbol{n}$ is a unit normal to streamline.

Passing to the limit $\mu\to\infty$ and taking into account that $\Omega\to\Omega(\Psi)$, from (28) we obtain

$$\Gamma(\Psi)\frac{d\Omega}{d\Psi}=1/3\Gamma(\Psi)+2/3S(\Psi)\Omega(\Psi), \tag{29}$$

where $S(\Psi)=\iint d\xi\,d\eta$ is the area of the region bounded by the streamline and $\Gamma(\Psi)=\iint\Omega\,d\xi\,d\eta$ is the circulation on the streamline; here integration is performed over the region defined by the closed streamline.

Thus, in the limit $\mu\to\infty$, the determination of the vortex pair pattern is reduced to the problem of sewing together [6] the potential inviscid flow (in the outer region) and the vortex (in inner region) subject to the condition of continuity of Ψ and $\nabla\Psi$ on the boundary. The form of the function $\Omega(\Psi)$ in this case is given by the ordinary (with respect to the variable Ψ) differential equation (29).

It can be shown that the assumption of boundedness of the limiting solution leads to its continuity, and, hence, Eq. (29) should be supplemented by the boundary condition $\Omega(0)=0$.

A numerical solution of the formulated problem was obtained together with L. Ya. Rybak. In the

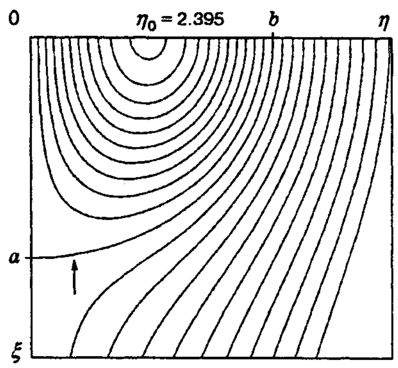

Fig. 1

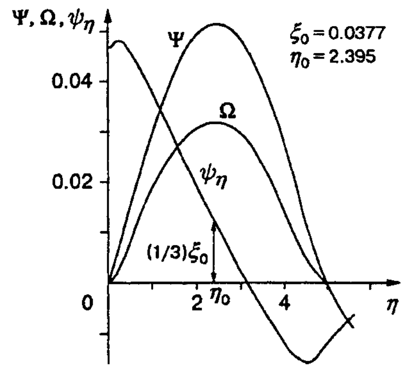

Fig. 2

limiting case $\mu \to \infty$, it is considerably simpler than the initial problem and gives the following results. Figure 1 shows the level lines of the stream function. The arrow denotes the curve $\Psi = 0$. The difference in values between two neighboring curves is constant ($\Delta\Psi = 0.05$). The maximum value $\Psi_{\max} = 0.0513$ is reached on the curve $\xi = 0$ at $\eta_0 = 2.395$. The ratio of the semiaxes is $b/a = 1.057$, where $b = 4.996$ and $a = 4.727$. Figure 2 shows the distribution of Ψ and Ω, and the vertical velocity component ψ_η on the curve $\xi = 0$ versus η in a fixed reference system. In this system, the vortex moves with a velocity $\xi_0/3$, where $\xi_0 = 0.0378$. The velocity and vortex momentum per unit length of the vortex pair can be written as

$$U = k_1 \frac{\Gamma}{R}, \quad 2j = k_2 \Gamma R,$$

where $k_1 = 0.070$, $k_2 = 0.972$, and R is half the distance between the centers of the vortex lines.

It should be noted that in the model considered, the calculation of the vorticity and velocity field in the limiting case of vanishing viscosity does not contain any empirical constant. The results are uniquely determined under the assumption that the turbulent viscosity does not depend on space coordinates. There are, however, qualitative experimental evidence and theoretical considerations that indicate that the turbulent viscosity is actually not constant over the volume of the vortex but decreases with approach to the cores of the vortex pair because of suppression of turbulence in the vortex core [2, 3, 5]. Such behavior of turbulent viscosity follows from the behavior of an additive transferred by a vortex ring [9]. However, at present, there are no experimental data, reliable model, or well-founded theoretical considerations that permit the spatial structure of this quantity to be adequately determined.

The value of α can be measured experimentally, and for large values of μ, it is asymptotically related to the calculated quantities by

$$\alpha = \frac{\eta_0}{\mu \xi_0} = \frac{63.5}{\mu}.$$

6. Comparison with Experiments. From formulas (7) and the calculations performed it follows that the maximum vertical velocity $u_{\max}$ is reached practically on the axis of the vortex pair and, according to self-similarity, it varies with distance as

$$u_{\max} = u_*(z_*/z)^{2/3},$$

where u_* is the value of this quantity at the point $z_* > z_s$ (z_* is the distance from the virtual origin of the vortex pair), which can be any point in the region where the self-similar regime is assumed. We set $l = z - z_*$. Then, the relation $(u_*/(u_{\max})^{3/2} - 1)/l = \text{const}$ should hold.

The experimental results of [2] agree well with the self-similar solution for $u_* = 6$ m/sec and $z_* = 613$ m (the data denoted in Fig. 11 of [2] by squares covers the maximum range of downstream distances).

From the aforesaid, the maximum velocity is related to the distance by

$$u_{\max} = W_0 Q \eta_0^2 u_m / [1 + \eta_0^2 \xi_0 \alpha Q (l/b)]^{2/3}, \tag{30}$$

where b is the wing span, $Q = P/2\rho W_0^2 b^2$, $u_m = \max \psi_\eta$ is the maximum value of this quantity on the curve $\xi = 0$, and l is the distance reckoned downstream from the point located at 40–50 wing spans from the moving body in the z direction. In this formula, all quantities, except for α, are determined from the formulation and solution of the self-similar problem. The value of α can be measured experimentally. However, we were unable to find literature values for this quantity. An analogy to vortex rings and some additional considerations suggest that the values of α are in the range 10^{-1}–10^{-2}. As noted above, α depends on the conditions of formation of vortex lines. In particular, attempts to decrease the vortex-line intensity by mounting antivortex generators of various types on wings can influence, from the viewpoint of the model considered, only the value of α. The calculation performed within the limit of vanishing viscosity gives $\eta_0^2 u_m = 0.28$ and $\eta_0^2 \xi_0 = 0.22$ and formula (30) allows one to evaluate the maximum velocity in the vortex wake.

A comparison of the flow structure found numerically with the one observed in experiments is impossible because of the lack of measurement results in the wake far region. If the structure obtained at rather long distances is extrapolated to the wake far region, considerable disagreement with the measurement results is observed: the calculated vorticity profile is smeared; the maximum vertical velocity is reached practically on the axis of the vortex pair, and it is considerably smaller than the one observed in experiments. This disagreement is apparently caused by the above-mentioned neglect of the decreased turbulent viscosity in the vortex core. It is possible that allowance for this circumstance by selection of a more complex model will allow one to eliminate the indicated disagreement and to approach the calculated values in (30) to experimental results. It should be noted, however, that in real situations, atmospheric stratification, which almost always takes place, has a significant influence on the evolution of the vortex pair and can lead to substantial departure from self-similarity.

Conclusion. The main result from the approach used in the present paper is as follows: the turbulent vortex pair formed past a moving heavy body is described as a first approximation by the self-similar solution of the corresponding equations beginning with a certain distance from the body and as long as the turbulent viscosity far exceeds the molecular viscosity. In real situations, this description is likely valid at distances about several kilometers downstream from the moving body. Self-similarity determines the position and intensity of vortex lines, and, thus, it becomes possible to evaluate the hazard to aircraft falling in the region of wake vortices.

The form of self-similarity, provided that molecular viscosity can be ignored, does not depend on the selection of a particular model for describing turbulent fluid flow. In any adequate model of any high level in the approximation considered, the law of conservation of vortex momentum per unit length should be fulfilled and invariant solutions with respect to the extension group [10] that correspond to the self-similarity obtained in the present paper should exist. Therefore, use of models of higher levels to determine the vortex-line pattern and the velocity field do not influence the form of self-similarity. At present, however, because of difficulties in conducting corresponding experiments, there are no fairly full and reliable measurements in the wake far region that can validate a particular model of high order. For this reason, in the present paper, we used a simplified model of turbulent fluid flow. At the same time, in view of the disagreement between the calculated vorticity-distribution pattern and experimental results, further studies in this direction are required.

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